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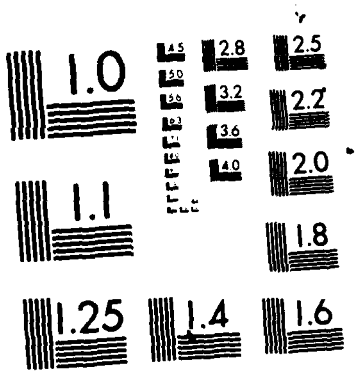
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PARALLEL SUCCESSIVE OVERRELAXATION METHODS FOR SYMMETRIC LINEAR COMPLEMENTARITY PROBLEMS AND LINEAR PROGRAMS

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ABSTRACT

A parallel successive overrelaxation (SOR) method is proposed for the solution of the fundamental symmetric linear complementarity problem. Convergence is established under a relaxation factor which approaches the classical value of 2 for a loosely coupled problem. The parallel SOR approach is then applied to solve the symmetric linear complementarity problem associated with the least norm solution of a linear program.

AMS (MOS) Subject Classifications: 90C25, 90C05

Key Words: Parallel algorithms, linear programming, complementarity problem, successive overrelaxation

Work Unit Number 5 (Optimization and Large Scale Systems)

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SIGNIFICANCE AND EXPLANATION

A framework for the parallel solution of very large sparse linear programs and symmetric linear complementarity problems is proposed. Linear programs with 80,000 variables and 20,000 constraints have been solved with the serial version of the proposed methods. Considerably larger problems will be tackled with the parallel algorithms.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

PARALLEL SUCCESSIVE OVERRELAXATION METHODS FOR SYMMETRIC LINEAR COMPLEMENTARITY PROBLEMS AND LINEAR PROGRAMS

O. L. Mangasarian[1] and R. De Leone[2]

1. Introduction

The purpose of this work is to propose a parallel successive overrelaxation (SOR) method for the solution of the fundamental symmetric linear complementarity problem: Find a z in the n-dimensional real space R^n such that

$$Mz + q \geq 0,\ z \geq 0,\ z(Mz + q) = 0 \tag{1}$$

where M is an $n \times n$ real symmetric matrix and q is a vector in the n-dimensional real space R^n. It is well known that (1) is a necessary optimality condition for the optimization problem

$$\min_{z \geq 0}\ f(x) := \min_{z \geq 0}\ \frac{1}{2} zMz + qz \tag{2}$$

and that (1) is a sufficient optimality condition for (2) if M is positive semidefinite. Part of the importance of this problem stems from the fact that the solution of very large sparse linear programs by successive overrelaxation methods (Refs. 1-3) can be reduced to problem (2) with a positive semidefinite symmetric matrix M. Until recently (Ref. 4) only serial methods had been proposed for the solution of (1). A primary result of this work is the convergence of a parallel SOR procedure for the solution of (1) under a relaxation stepsize $\omega \in (0, \frac{2}{1+\sigma})$ given in (15) below, where σ is a positive number depending on the coupling that exists between the row blocks of the matrix M that are being coprocessed by the parallel SOR procedure. If the row blocks are loosely coupled, then σ is small. In fact σ is zero for uncoupled row blocks which results in $\omega \in (0, 2)$, the classical interval for the serial SOR procedure (Ref. 5). We give now a brief summary of the paper.

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We begin in Section 2 with some known preliminary results for the serial SOR algorithm for solving (1). A key role will be played by Algorithm 2.1 below which is a special case of Algorithm 2.1 of Ref. 1. Although Algorithm 2.1 was originally conceived as a serial SOR algorithm, it turns out that it is general enough to also encompass a parallel SOR scheme. The parallel SOR is achieved by taking a decomposition for the matrix M other than the standard $L+D+U$ decomposition, where L is the strictly lower triangular part, D is the diagonal part and U is the strictly upper triangular part of M. The rest of Section 2 of the paper is concerned with a useful modification of Algorithm 2.1 (Algorithm 2.3) and with convergence results for both algorithms. In Section 3, which constitutes the core of the paper, we construct parallel SOR algorithms for the solution of (1) by appropriately decomposing the matrix M so that it satisfies the assumptions of Algorithm 2.1 and such that parallel computation of the iterates is possible. In particular we decompose the matrix M into k blocks of disjoint row matrices and apply one sweep of the serial SOR Algorithm 2.1 to each block simultaneously. After this sweep the information is shared among the k blocks and the process is repeated. Under an appropriate stepsize condition we show in Theorem 3.5 that each accumulation point of such a parallel SOR algorithms solves (1). Theorem 3.6 establishes convergence of all the iterates of parallel SOR algorithms under the assumption that $Mz + q > 0$ has a solution. In Section 4 we apply the parallel SOR algorithms of Section 3 to finding the least 2-norm solution of a linear program, which turns out to be precisely a problem of the form (2) with a positive semidefinite matrix. Theorem 4.2 establishes the linear convergence of parallel SOR schemes for the least 2-norm solution of a linear program. We conclude the paper in Section 5 with some brief remarks regarding computational implementation of the proposed parallel schemes.

We briefly describe our notation now. For a vector x in the n-dimensional real space R^n, x_+ will denote the vector in R^n with components $(x_+)_i = \max\{x_i, 0\}$, $i = 1,...,n$. The scalar product of two vectors x and y in R^n will be simply denoted by xy. For $1 \leq p \leq \infty$, the p-norm $\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ of a vector in R^n will be denoted by $\|x\|_p$. R^n_+ will denote the nonnegative orthant or the set of points in R^n with nonnegative components, while $R^{m \times n}$ will denote the set of all $m \times n$ real matrices. For $A \in R^{m\times n}$, A^T will denote the transpose, A_i will denote the ith row, A_{ij} the element in row i and column j, and for $I \subset \{1,...,m\}$, $J \subset \{1,...,n\}$, A_I will denote the submatrix of A with rows A_i, $i \in I$,

while A_{IJ} will denote the submatrix of A with elements A_{ij}, $i \in I$, $j \in J$. Similarly for $x \in R^n$ and $I_\ell \subset \{1, ..., n\}$, x_{I_ℓ} will denote x_i, $i \in I_\ell$. The set $\{I_1, I_2, ..., I_K\}$ is said to be a consecutive partition of $\{1, ..., n\}$ if it is a partition of $\{1, ..., n\}$ such that $i < j$ for $i \in I_\ell$, $j \in I_{\ell+1}$ and $\ell = 1, ..., k-1$. For a twice differentiable function $\theta: R^m \times R^n \to R$, $\nabla^2\theta$ will denote the $(m+n) \times (m+n)$ Hessian matrix, while $\nabla\theta$ will denote the $m+n$ gradient vector and $\nabla\theta(0, 0) := \nabla\theta(u, v)|_{u=0,\, v=0}$. Here and throughout the symbols $:=$ and $=:$ denote definition of the term on the left and right sides respectively.

2. Preliminary Background

In this section we give some background results needed to derive our parallel SOR algorithm for solving the symmetric linear complementarity problem and linear programs. We begin first with a special case of the serial SOR Algorithm 2.1 of Ref. 1.

Algorithm 2.1 (Serial SOR algorithm for (1)) Let $z^0 \geq 0$. For $i = 0, 1, 2, \ldots$, let

$$z^{i+1} = \left(z^i - \omega E^i\left(Mz^i + q + K^i(z^{i+1} - z^i)\right)\right)_+ \tag{3}$$

where $\omega > 0$, and $\{E^i\}$ and $\{K^i\}$ are bounded sequences of matrices in $R^{n\times n}$, with each E^i being a positive diagonal satisfying

$$E^i > \alpha I \tag{4}$$

for some $\alpha > 0$ and such that for some $\gamma > 0$

$$y\left((\omega E^i)^{-1} + K^i - M/2\right)y \geq \gamma \|y\|_2^2, \ \forall i, \ \forall y \in R^n \tag{5}$$

Remark 2.2 If we let

$$L + D + U := M \tag{6}$$

where L is the strictly lower triangular part of M, D is the diagonal of M and U is the strictly upper triangular part of M, then the iteration (3) is an explicit one if we set $K^i = L$, U or 0. More specifically for $K^i = L$ or U and $E^i = D^{-1}$ (assuming that D is positive) iteration (3) gives the projected SOR algorithm studied in Ref. 1 and condition (5) becomes the familiar SOR relaxation factor condition: $0 < \omega < 2$. When $K^i = 0$ we have the projected Jacobi method (Ref. 1). However K^i may be <u>any</u> matrix as long as (5) is satisfied, in which case the iteration (3) can be considered as solving a (hopefully simpler) linear complementarity problem.

An important useful modification of Algorithm 2.1 has been proposed by Subramanian (Ref. 6) in which instead of taking z^{i+1} of (3), any other point in the nonnegative orthant is taken with a value of f not exceeding $f(z^{i+1})$. This leads to the following algorithm.

Algorithm 2.3 (Modified serial SOR algorithm for (1)) Let $z^0 \geq 0$. For $i = 0, 1, 2, \ldots$, let

$$s^i = \left(z^i - \omega E^i\left(Mz^i - q - K^i(s^i - z^i)\right)\right)_+ \tag{7}$$

where $\omega > 0$ and the sequences of matrices $\{E^i\}$ and $\{K^i\}$ satisfy all the requirements of Algorithm 2.1. Choose $z^{i+1} \geq 0$ such that $f(z^{i+1}) \leq f(s^i)$.

The important point to note about Algorithm 2.3 is that is allows a whole class of algorithms to be based on Algorithm 2.1. Typically, z^{i+1} of Algorithm 2.3 is obtained from s^i by some sort of line search.

The convergence of Algorithm 2.1 was establsihed in Theorem 2.1 of Ref. 1, and the convergence of Algorithm 2.3 in Theorem 4.2 of Chapter 3 of Ref. 6. We combine these results into the following fundamental convergence result under no assumption on the matrix M other than symmetry.

Theorem 2.4 (Convergence of serial SOR algorithms for (1)) Let M be symmetric. Each accumulation point of the sequence $\{z^i\}$ of Algorithm 2.1 or 2.3 solves the linear complementarity problem (1).

Note that Theorem 2.4 does not guarantee the existence of an accumulation point for the sequence $\{z^i\}$. To do that we need additional assumptions (Ref. 1, Theorem 2.2) such as the following.

Theorem 2.5 (Strong convergence of serial SOR algorithms for (1)) Let M be symmetric and positive semidefinite, and let

$$Mz + q > 0 \quad \text{for some} \quad z \in R^n \tag{8}$$

Then, the sequences $\{z^i\}$ of Algorithms 2.1 and 2.3 are bounded and have accumulation points. Each accumulation point of $\{z^i\}$ solves (1).

With this background material we are prepared to introduce our parallel SOR algorithm for solving the symmetric linear complementarity problem (1).

3. Parallel SOR for the Symmetric Linear Complementarity Problem

The key idea of our approach here is to consider K^i of Algorithm (2.1) as a substitution operator which replaces the old data z^i by the new data z^{i+1}. If for example $K^i := L$, where L is the strictly lower triangular part of the <u>whole</u> matrix M, then z_j^{i+1} replaces z_j^i during the computation of z_ℓ^{i+1} for <u>all</u> $\ell > j$. Now, consider instead the following procedure. Break M into k blocks of rows as follows:

$$M =: \begin{bmatrix} M_{I_1} \\ M_{I_2} \\ \vdots \\ M_{I_k} \end{bmatrix} \tag{9}$$

where the blocks M_{I_j} correspond to the variables z_{I_j} and $\{I_1, I_2, ..., I_k\}$ is a consecutive partition of $\{1,2,...,n\}$. Now partition M_{I_j} as follows

$$M_{I_j} =: \begin{bmatrix} M_{I_j I_j} & M_{I_j \bar{I}_j} \end{bmatrix} \tag{10}$$

where $\bar{I}_j$ is the complement of I_j in $\{1,2,...,n\}$. Thus $M_{I_j I_j}$ is a principal square submatrix of M with elements M_{rs}, $r \in I_j$ and $s \in I_j$. We further partition $M_{I_j I_j}$ as follows

$$M_{I_j I_j} =: L_{I_j I_j} + D_{I_j I_j} + U_{I_j I_j} \tag{11}$$

where $L_{I_j I_j}$ is the strictlly lower triangular part of $M_{I_j I_j}$, $D_{I_j I_j}$ its diagonal part and $U_{I_j I_j}$ its strictly upper triangular part. Thus for example if $k = 3$ we would have the following decomposition of M

$$M = \begin{bmatrix} M_{I_1} \\ M_{I_2} \\ M_{I_3} \end{bmatrix} = \begin{bmatrix} M_{I_1 I_1} & M_{I_1 I_2} & M_{I_1 I_3} \\ M_{I_2 I_1} & M_{I_2 I_2} & M_{I_2 I_3} \\ M_{I_3 I_1} & M_{I_3 I_2} & M_{I_3 I_3} \end{bmatrix} \tag{12}$$

Now let K^i of Algorithm 2.1 be defined by a <u>block diagonal</u> matrix as follows

$$K^i = K := \begin{bmatrix} L_{I_1 I_1} & & & \\ & L_{I_2 I_2} & & \\ & & \ddots & \\ & & & L_{I_k I_k} \end{bmatrix} \tag{13}$$

where each $L_{I_j I_j}$ is a strictly lower triangular matarix defined in (11).

Algorithm 2.1 can now be performed for each row block I_j, $j = 1, \ldots, k$, simultaneously, that is in parallel. Note that this is <u>not</u> a block Jacobi iteration. More specifically we have the following algorithm.

Algorithm 3.1 (Parallel SOR for (1)) Let $\{I_1, I_2, \ldots, I_k\}$ be a consecutive partition of $\{1, 2, \ldots n\}$, let the diagonal D of M have positive elements and let $z^0 \geq 0$. For $i = 0, 1, 2, \ldots$, let

$$z_{I_j}^{i+1} = \left(z_{I_j}^i - \omega D_{I_j I_j}^{-1}\left(M_{I_j} z^i + q_{I_j} + L_{I_j I_j}(z_{I_j}^{i+1} - z_{I_j}^i)\right)\right)_+ \qquad j = 1, \ldots, k \tag{14}$$

where

$$0 < \omega < \min_{1 \leq j \leq k} \min_{\ell \in I_j} \frac{2}{1 + \sum_{s \in \bar{I}_j} |M_{\ell s}| / D_{\ell\ell}} \tag{15}$$

Remark 3.2 Iteration (14) can be performed in parallel on k processors. The new value z^{i+1} must then be shared between the k processors.

Remark 3.3 If all $M_{I_j \bar{I}_j}$ are zero then $0 < \omega < 2$ for all j, which is the standard SOR relaxation factor range. This corresponds to k uncoupled linear complementarity problems. If all $M_{I_j \bar{I}_j}$ are small relative to D_{I_j}, which corresponds to a loosely coupled linear complementarity problem, the upper bound on ω given by (15) is close to 2.

We state now a parallel SOR version of Algorithm 2.3.

Algorithm 3.4 (Modified parallel SOR for (1)) Let the assumptions of Algorithm 3.1 hold. For $i = 0, 1, 2, \ldots$, let

$$s_{I_j}^i = \left(z_{I_j}^i - \omega D_{I_j I_j}^{-1}\left(M_{I_j} z^i + q_{I_j} + L_{I_j I_j}(s_{I_j}^i - z_{I_j}^i)\right)\right)_+ \qquad j = 1, \ldots, k \tag{16}$$

where ω satisfies (15). Chose $z^{i+1} \geq 0$ such that $f(z^{i+1}) \leq f(s^i)$.

We can establish the convergence of Algorithms 3.1 and 3.4 by appealing to Theorem 2.4. We have then the following convergence result.

Theorem 3.5 (Convergence of parallel SOR algorithms for (1)) Let M be symmetric. Each accumulation point of the sequence $\{z^i\}$ of Algorithms 3.1 and 3.4 solves the linear complementarity problem (1).

Proof By Theorem 2.4 we only need to establish that condition (5) is satisfied by the choice (13) for K^i, $E^i := D^{-1}$ and ω satisfying assumption (15). We have

$$
\begin{aligned}
& y((\omega E^i)^{-1} + K^i - M/2)y \\
&= \frac{1}{2} y(2\omega^{-1}D + 2K - M)y \\
&= \frac{1}{2}\sum_{r=1}^{k} y_{I_r}(2\omega^{-1}D_{I_r} + 2K_{I_r} - M_{I_r})y \\
&= \frac{1}{2}\sum_{r=1}^{k} y_{I_r}\left[(2\omega^{-1}D_{I_rI_r} + 2K_{I_rI_r} - M_{I_rI_r})y_{I_r} - \sum_{s\neq r} M_{I_rI_s} y_{I_s}\right] \\
&= \frac{1}{2}\sum_{r=1}^{k} y_{I_r}\left[(2\omega^{-1}D_{I_rI_r} + 2L_{I_rI_r} - (L_{I_rI_r} + D_{I_rI_r} + U_{I_rI_r}))y_{I_r} - \sum_{s\neq r} M_{I_rI_s} y_{I_s}\right] \\
&= \frac{1}{2}\sum_{r=1}^{k} y_{I_r}\left[(2\omega^{-1} - 1)D_{I_rI_r} y_{I_r} - \sum_{s\neq r} M_{I_rI_s} y_{I_s}\right] \\
&\qquad\qquad\qquad \text{(Because } L = U^T\text{)} \\
&= \frac{1}{2}\sum_{r=1}^{k} y_{I_r}\begin{bmatrix}(2\omega^{-1} - 1)D_{I_rI_r} & -M_{I_r\bar{I}_r}\end{bmatrix}\begin{bmatrix} y_{I_r} \\ y_{\bar{I}_r}\end{bmatrix} \\
&= \frac{1}{2}[\,y\,]\begin{bmatrix}(2\omega^{-1} - 1)D_{I_1I_1} & & -M_{I_1\bar{I}_1} \\ & \ddots & \\ -M_{I_k\bar{I}_k} & & (2\omega^{-1} - 1)D_{I_kI_k}\end{bmatrix}\begin{bmatrix} y \end{bmatrix} \geq \gamma\|y\|_2^2
\end{aligned}
$$

where last inequalilty holds for some $\gamma > 0$ because of the positive definiteness of the symmetric $n \times n$ matrix (preceding the inequality) which is induced by its row diagonal dominance (Ref. 5). The row diagonal dominance is precisely a consequence of assumption (15). ∎

Having establilshed that the parallel SOR Algorithms 3.1 and 3.4 can be considered as special cases of the general serial Algorithms 2.1 and 2.3 respectively, the following strong convergence result is a direct consequence of Theorem 2.5.

Theorem 3.6 (Strong convergence of parallel SOR algorithms for (1)) Let M be symmetric and positive semidefinite and let assumption (8) hold. Then the sequences $\{z^i\}$ of

Algorithms 3.1 and 3.4 are bounded and have accumulation points. Each accumulation point of $\{z^i\}$ solves (1).

Remark 3.7 Minor changes in the proof of Theorem 3.5 allows us to have a different ω_j for each $j = 1, \ldots, k$ in (14). In particular all we need is that for $j = 1, \ldots, k$, ω_j must satisfy

$$0 < \omega_j < \min_{\ell \in I_j} \frac{2}{1 + \sum_{s \in \bar{I}_j} |M_{\ell s}| / D_{\ell \ell}} \tag{17}$$

This results in larger stepsizes for Algorithms 3.1 and 3.4.

We now turn our attention to the parallel solution of linear programs.

4. Parallel Solution of Linear Programs

The key idea here is to find the least 2-norm solution of a linear program by converting the problem to a positive semidefinite linear complementarity problem (Refs. 2-3) and to use the parallel SOR procedures proposed in the previous section. We will present a parallel implementation here of the linearlly convergent iterative scheme proposed in Ref. 7.

We consider the linear program

$$\min_{x} \; cx \quad \text{subject to } Ax \geq b, \; x \geq 0 \tag{18}$$

where $c \in R^h$, $b \in R^m$ and $A \in R^{m \times h}$, and its dual

$$\max_{u} \; bu \quad \text{subject to } A^T u \leq c, \; u \geq 0 \tag{19}$$

It is known (Refs. 2-3) that $\bar{x}$ is the unique least 2-norm solution to (18) if and only if $\bar{x}$ is the unique solution to the quadratic program

$$\min_{x} \; cx + \frac{\varepsilon}{2} xx \quad \text{subject to } Ax \geq b, \; x \geq 0 \tag{20}$$

for all $\varepsilon \in (0, \bar{\varepsilon}]$ for some $\bar{\varepsilon} > 0$. The dual to the quadratic program (20) is

$$\max_{x,u,v} - \frac{\varepsilon}{2} + bu \quad \text{subject to } v = \varepsilon x - A^T u + c, \; (u,v) \geq 0 \tag{21}$$

To solve (20) for a fixed positive ε we shall use the parallel SOR procedures of Section 3 applied to its dual (21) with the variable x eliminated through the dual constraint

$$x = (A^T u + v - c)/\varepsilon \tag{22}$$

and thus obtaining the dual problem

$$\min_{(u,v) \geq 0} \theta(u,v) := \min_{(u,v) \geq 0} \frac{1}{2} \|A^T u + v - c\|_2^2 - \varepsilon bu \tag{23}$$

which is precisely of the form (2) with a positive semidefinite matrix and hence is equivalent to the symmetric linear complementarity problem (1) on R^{m+h} with $M := \nabla^2 \theta(u,v)$ and $q := \nabla \theta(0,0)$. We shall now describe a linearly convergent sequential parallel SOR

procedure for solving (23) based on Ref. 7 and the results of Section 3. We first need a definition.

Definition 4.1 (Approximate solutions to (23) and (20)) For a fixed positive ε <u>any</u> point in R_+^{m-h} is an <u>approximate</u> solution to the dual quadratic program (23) and is designated by $(u(\varepsilon), v(\varepsilon))$. The corresponding $x(\varepsilon)$ in R^h defined by (22) with $(u, v) = (u(\varepsilon), v(\varepsilon))$ is an <u>approximate</u> solution to the quadratic program (20). The <u>residual</u> $r(\varepsilon)$ associated with $(u(\varepsilon), v(\varepsilon), x(\varepsilon))$ is defined by

$$r(\varepsilon) := \left[|x(\varepsilon)v(\varepsilon) + u(\varepsilon)(Ax(\varepsilon) - b)| + \|(b - Ax(\varepsilon))_+\|_\infty + \|(-x(\varepsilon))_+\|_\infty \right]^{1/2} \tag{24}$$

Note that for an $\varepsilon > 0$ and an approximate solution $(u(\varepsilon), v(\varepsilon))$ to (23) and a corresponding approximate solution $x(\varepsilon)$ to (20), $r(\varepsilon) = 0$ if and only if $(u(\varepsilon), v(\varepsilon))$ is an exact solution of (23) and $x(\varepsilon)$ is the unique exact solution of (20).

We are prepared now to state and prove a linearly convergent parallel SOR procedure for computing the least 2-norm solution of the linear program (18).

Theorem 4.2 (Linearly convergent parallel SOR for linear programs) Assume that the linear program (18) is solvable and that $b \neq 0$. Let $\{\varepsilon_0, \varepsilon_1, \ldots\}$ be a decreasing sequence of positive numbers such that

$$\varepsilon_{i+1} = \mu\varepsilon_i \quad \text{for some} \quad \mu \in (0,1) \tag{25}$$

and let $\{u(\varepsilon_i), v(\varepsilon_i), x(\varepsilon_i)\}$ be a corresponding sequence of approximate solutions to (23) and (20) satisfying Defifnition 4.1 and obtained by either of the parallel SOR Algorithms 3.1 or 3.4 applied to (23) and such that their residuals as defined by (24) satisfy

$$r(\varepsilon_{i+1}) \leq \nu r(\varepsilon_i) \tag{26}$$

for some $\nu \geq 0$ such that

$$\nu < \mu^{1/2} \tag{27}$$

Then the sequence $\{x(\varepsilon_i)\}$ *converges to* $\bar{x}$, the least 2-norm solution of the linear program (18) at the linear root rate

$$\|x(\varepsilon_i) - \bar{x}\|_2 \leq \delta(\nu/\mu^{1/2})^i \quad \text{for} \quad i \geq \bar{i} \tag{28}$$

for some constant δ and some integer $\bar{i}$.

Proof See Theorem 3.7 of Ref. 7. ∎

5. Conclusion

We have presented a framework for the parallel solution of symmetric linear complementarity problems and linear programs. The proposed SOR algorithm is best suited for a tightly-coupled shared-memory multiprocessor such as the one to be acquired by the Computer Sciences Department at Madison. However we plan to test the proposed algorithm on the existing loosely-coupled 20-processor token-ring-connected Crystal machine (Ref. 8) of the Computer Sciences Department in order to develop efficient computational implementations of our algorithm. Because we have been able to solve sparse linear programs of size 20,000 variables and 5,000 constraints by the serial version of our SOR procedure in 78 minutes on a VAX 11/780 (Ref. 2), we are hopeful of solving substantially larger problems by our parallel approach.

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A parallel successive overrelaxation (SOR) method is proposed for the solution of the fundamental symmetric linear complementarity problem. Convergence is established under a relaxation factor which approaches the classical value of 2 for a loosely coupled problem. The parallel SOR approach is then applied to solve the symmetric linear complementarity problem associated with the least norm solution of a linear program.

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